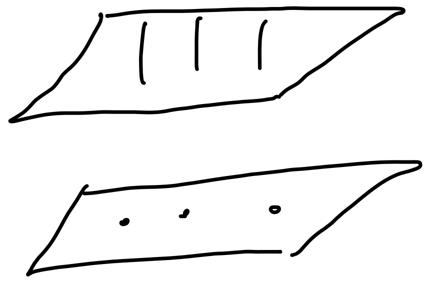
$B = V\left(\{Y_{i1}(X_2 - a_{i2}X_3) - Y_{i2}(X_1 - a_{i1}X_3) \mid i=1,\dots,r\}\right) \subseteq \mathbb{P}^2 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$

$$\begin{tikzcd} B \arrow[d, "\pi"] \\ \mathbb{P}^2 \end{tikzcd}$$



study the behavior of π around some pt $Q \in E_i$.

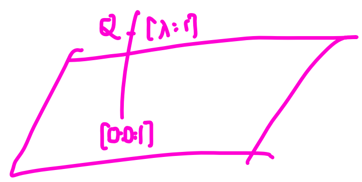
Fact: locally $\pi : B \to \mathbb{P}^2$ looks like $\psi : \mathbb{A}^2 \to \mathbb{A}^2$ (in §7.2)

i.e. $\forall\, Q \in B$ $\exists\, V' \overset{\text{open}}{\hookrightarrow} B$ & $W' \overset{\text{open}}{\hookrightarrow} \mathbb{A}^2$ s.t.

$$\begin{tikzcd}
\mathbb{A}^2 \arrow[d, "\psi"'] \arrow[dr, phantom, "\circlearrowleft"] & W' \arrow[l, hook', "\text{open}"'] \arrow[r, "\varphi", "\cong"'] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & V' \arrow[r, hook, "\text{open}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "\pi"] \\
\mathbb{A}^2 & W \arrow[l, hook', "\text{open}"'] \arrow[r, "\varphi_3", "\cong"'] & V \arrow[r, hook, "\text{open}"] & \mathbb{P}^2
\end{tikzcd}$$

Pf: WLOG: $i=1$ and $P_1 = [0:0:1]$ $\quad Q = [\lambda:1] \in \mathbb{P}^1_i$ $\quad \lambda \in k$

$\varphi_3 : \mathbb{A}^2 \overset{\cong}{\Rightarrow} U_3 \overset{\text{open}}{\hookrightarrow} \mathbb{P}^2 \quad (x,y) \mapsto [x:y:1]$



$V := U_3 \setminus \{P_2, \dots, P_r\} \ni P_1$

$W := \varphi_3^{-1}(V) \subset \mathbb{A}^2$

$\psi : \mathbb{A}^2 \to \mathbb{A}^2 \quad (x,z) \mapsto (x, xz)$

$W' := \psi^{-1}(W)$

$\varphi : W' \to \mathbb{P}^2 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$

$(x,z) \mapsto \left([x:xz:1], [1:z], f_2([x:xz:1]), \dots, f_r([x:xz:1])\right)$

Then φ is a morphism with image

$$V' = B \setminus \left(\bigcup_{i>1} E_i \cup V(X_3) \cup V(Y_{11}) \right) \ni Q$$

and satisfies $\pi \circ \varphi = \varphi_3 \circ \psi$.

$$\begin{tikzcd}
W' \arrow[rr, "\varphi"] \arrow[dr] \arrow[dd, "\psi"'] & & \mathbb{P}^2 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \arrow[dd, "\pi"] \\
& V' \hookrightarrow B \arrow[ur, hook] & \\
W \arrow[rr, "\varphi_3"] & & \mathbb{P}^2
\end{tikzcd}$$

The inverse morphism of φ is (the restriction to V' of)

$$\mathbb{P}^2 \times \cdots \times \mathbb{P}^1 \setminus V(X_3 Y_{11}) \longrightarrow \mathbb{A}^2$$

$$([x_1 : x_2 : x_3], [y_{11} : y_{12}], \dots) \longmapsto (x_1/x_3, \ y_{12}/y_{11})$$

$C \subset \mathbb{P}^2$ irr curve.

$C_0 := C \cap U$, $\quad C_0' := \pi^{-1}(C_0) \subset G$

$C' :=$ closure of C_0' in B.

$\pi \rightsquigarrow f : C' \to C$ (birational morphism)

$$\begin{array}{ccc} C' & \to & C \\ \cup & & \cup \\ C_0' & \xrightarrow{\sim} & C_0 \end{array}$$

f looks like the affine map in §7.2.

Prop 1. C = irr. proj. plane curve.
Suppose all multiple pts of C are ordinary. Then
$\exists$ birational morphism $f: C' \to C$
C' nonsingular projective.

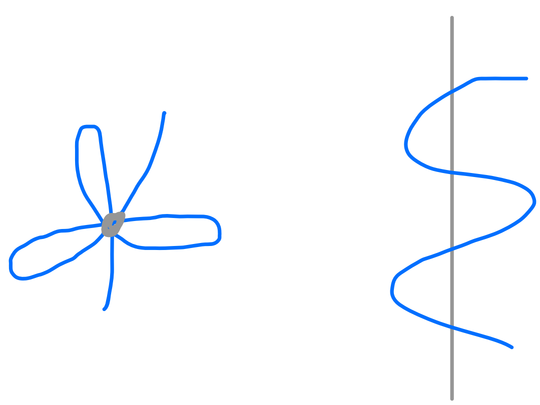
Pf: apply (8) $\Rightarrow f: C' \to C$ step 2 in § 7.2 $\Rightarrow C'$ nonsingular.

$\exists$ ordinary projective plane curve $\in [C]$

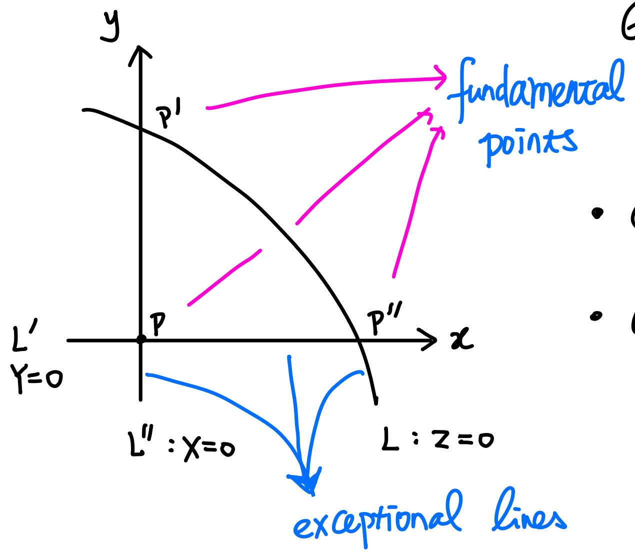
$\Rightarrow \exists$ nonsingular projective curve $\in [C]$.

The next question is: can we find an ordinary proj plane curve in $[C]$?

To answer this, we need study q. transformations.



standard quadratic transformation



$Q : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$

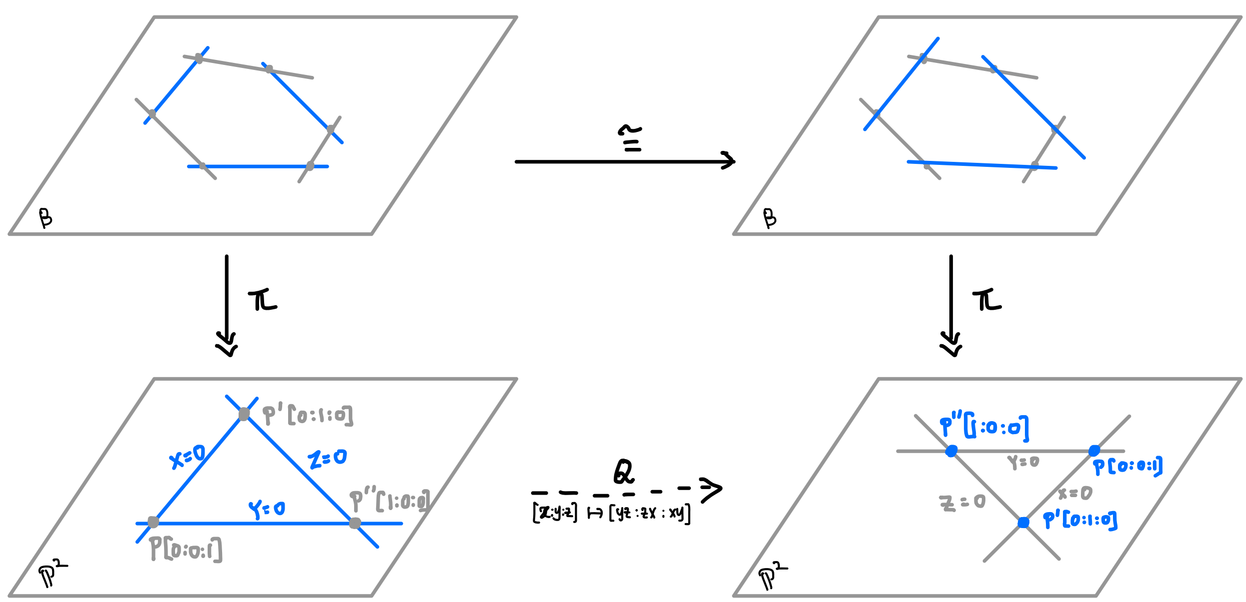
$[x:y:z] \longrightarrow [yz : xz : xy]$

- Q is defined on $\mathbb{P}^2 \setminus \{P, P', P''\}$
- $Q \circ Q = id$ on $V = \mathbb{P}^2 \setminus (L \cup L' \cup L'')$

On V, Q is also given by

$[x:y:z] \mapsto [\frac{1}{x} : \frac{1}{y} : \frac{1}{z}]$

$([x_1 : x_2, x_3], [y_{11}:y_{12}], [y_{21}:y_{22}], [y_{31}, y_{32}]) \longmapsto ([x_2x_3 : x_1x_3 : x_1x_2], ($



• $C \subseteq \mathbb{P}^2$ irr. curve ($C \neq$ exceptional line)

$\Rightarrow U \leftrightarrow C \cap U \hookrightarrow C$

$Q^{-1}(C\cap U) = Q(C\cap U) \hookrightarrow U$

• $C' :=$ closure of $Q^{-1}(C\cap U)$ in $\mathbb{P}^2$

$\Rightarrow \quad Q : C' \setminus \{P, P', P''\} \to C \quad \& \quad (C')' = C$.

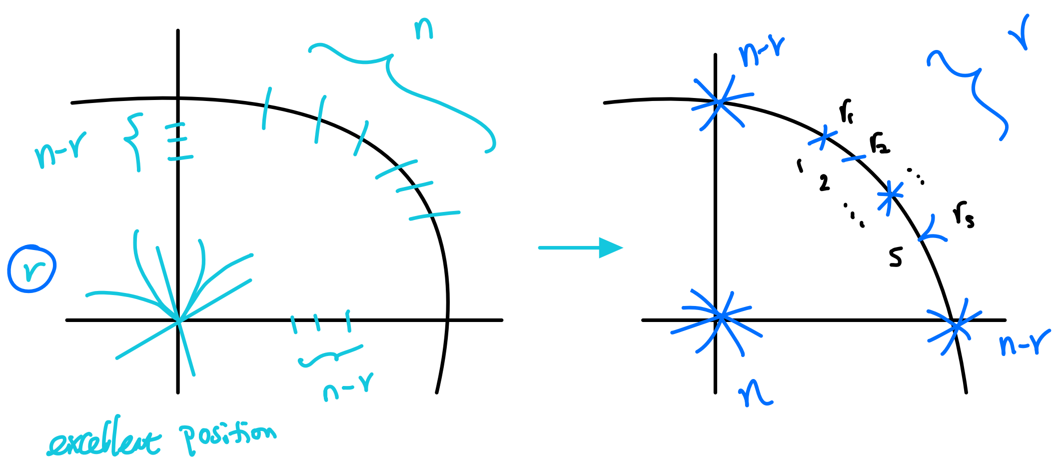
$F \in k[X,Y,Z] \qquad C = \{F = 0\} \qquad n = \deg(F)$

$F^Q := F(YZ, XZ, XY) \leftarrow$ form of deg $2n$.

algebraic transform of F

$m_P(C) = r, \quad m_{P'}(C) = r', \quad m_{P''}(C) = r'' \quad \Rightarrow \quad F^Q = Z^r Y^{r'} X^{r''} F'$

Fact: $V(F') = C'$



Fact: C in excellent position $\Rightarrow$ C' has following multiple pts

a) $C' \cap U \overset{1:1}{\longleftrightarrow} C \cap U$ (mult. pts)

preserves multiple pts, multiplicity, ordinary mult. pts.

b) P, P', P'' ordinary on C' with multiplication $n, n-r, n-r$

c). $\nexists$ non-fundamental pts on $C' \cap L'$ or on $C' \cap L''$

d). Let $\{P_1, \cdots, P_s\}$ = non-fundamental pts on $C' \cap L$ then

$$m_{P_i}(C') \leq I(P_i, C' \cap L)$$

$$\sum I(P_i, C' \cap L) = r$$

e) $g^*(C') := g^*(C) - \sum_{i=1}^{s} \frac{r_i(r_i-1)}{2}$ $\left(\text{where } g^*(C) := \frac{(n-1)(n-2)}{2} - \sum_{P \in C} \frac{r_P(r_P-1)}{2} \right)$

$r_i = m_{P_i}(C')$

§7.5 nonsingular models of curves

Thm3 C = proj curve. Then $\exists !$ nonsingular proj. curve X & birational morphism $f: X \twoheadrightarrow C$.

↖ nonsingular model of C

! means: if $f': X' \twoheadrightarrow C$ is another one, then

$$\begin{tikzcd} X \arrow[rr, "\exists \cong g"] \arrow[dr, "f"'] & \circlearrowleft & X' \arrow[dl, "f'"] \\ & C & \end{tikzcd}$$

pf: §7.1, cor 1 $\Rightarrow$ uniqueness.

existence:

cor in §6.6 $\Rightarrow$ plane curve $C' \overset{\text{birational}}{\dashrightarrow} C$

Thm2 §7.4 $\Rightarrow$ plane curve with only ordinary mult. pts. $C' \overset{\text{birational}}{\dashrightarrow} C$

Prop1 §7.3 $\Rightarrow$ nonsingular curve $X \overset{\text{birational}}{\dashrightarrow} C$

cor1 §7.1 $\Rightarrow$ $X \overset{\text{birational}}{\to} C$

surjective: 1° C = plane curve; Prob 7.13, Prop1 §7.3 $\Rightarrow X \twoheadrightarrow C$

2° general. $\exists\, C \overset{g}{\twoheadrightarrow} C_1$ (proj plane curve) s.t. $g^{-1}(g(P)) = \{P\}$ (Prob 6.43)